# 3 Modelling approach

The TYFIANT COED approach incorporates aspects of a mechanistic stand model developed by Wenk *et al.* (1990) and Wenk (1994), which are adapted to a spatially explicit individual tree approach.

## 3.1 Fundamental definitions

The modelling approach is based on two fundamental definitions.

**Definition 3.1.** The increment $i_Y$ is the difference in the value of a particular growth quantity $Y$ at different times $t$ and $t+\Delta t$:

(3.1) $$i_Y = Y_{i,t+\Delta t} - Y_{i,t}$$

**Definition 3.2.** The increment is the product of growth quantity $Y$ and relative increment $p$.

(3.2) $$i_Y = Y_{i,t+\Delta t} \cdot p_{i,t+\Delta t}$$

**Remark 3.1.** The index $i$ can denote an individual tree or a whole forest stand.

The two equations can now be set equal and be solved for $Y_{i,t+\Delta t}$.

$$Y_{i,t+\Delta t} - Y_{i,t} = Y_{i,t+\Delta t} \cdot p_{i,t+\Delta t} \qquad \Big|: Y_{i,t+\Delta t}$$

$$\Leftrightarrow \quad 1 - \frac{Y_{i,t}}{Y_{i,t+\Delta t}} = p_{i,t+\Delta t} \qquad \Big| + \frac{Y_{i,t}}{Y_{i,t+\Delta t}}$$

$$\Leftrightarrow \quad 1 = p_{i,t+\Delta t} + \frac{Y_{i,t}}{Y_{i,t+\Delta t}} \qquad \Big| - p_{i,t+\Delta t}$$

$$\Leftrightarrow \quad 1 - p_{i,t+\Delta t} = \frac{Y_{i,t}}{Y_{i,t+\Delta t}} \qquad \Big| 1:$$

$$\Leftrightarrow \quad \frac{1}{1 - p_{i,t+\Delta t}} = \frac{Y_{i,t+\Delta t}}{Y_{i,t}} \qquad \Big| \cdot Y_{i,t}$$

$$\Leftrightarrow \quad Y_{i,t+\Delta t} = Y_{i,t} \cdot \frac{1}{1 - p_{i,t+\Delta t}} \qquad (3.3)$$

The result of this transformation is equation (3.3), which states

*"The growth quantity at time $t+\Delta t$ is equal to the product of the growth quantity at time $t$ and the reciprocal of the relative increment subtracted from 1."*

**Definition 3.3.** The growth multiplier $M$ is defined as the term $\frac{1}{1 - p_{i,t+\Delta t}}$ with the index $t+\Delta t$:

(3.4) $$M_{i,t+\Delta t} = \frac{1}{1 - p_{i,t+\Delta t}}$$

With (3.4) in mind formula (3.3) can be written as:

$$\Leftrightarrow \qquad Y_{i,t+\Delta t} = Y_{i,t} \cdot M_{i,t+\Delta t} \qquad \Big| : Y_{i,t}$$

$$\Leftrightarrow \qquad \frac{Y_{i,t+\Delta t}}{Y_{i,t}} = M_{i,t+\Delta t}$$

This explains that the growth multiplier is defined as the ratio of a particular growth quantity at different times. It always refers to the end $t + \Delta t$ of the observation or forecasting period.

## 3.2 Function of the relative increment

$p_{i,t+\Delta t}$ is the relative increment at time $t + \Delta t$. From time series data where consecutive measurements of $Y$ are available $p_{i,t+\Delta t}$ can be calculated directly by means of formula (3.5) which uses volume in this particular example.

$$(3.5) \qquad p_{i,t+\Delta t} = \frac{V_{i,t+\Delta t} - V_{i,t}}{V_{i,t+\Delta t}}$$

Mathematically increment is the first derivative of a growth function. There are several functions available that have been successfully tested and deliver appropriate values for $p_{i,t+\Delta t}$ that can be inserted in formula (3.4). A growth function commonly used is Chapman/Richards:

$$(3.6) \qquad y = a \cdot \left(1 - e^{b \cdot t}\right)^{c} \text{ with the increment function } y' = a \cdot b \cdot c \cdot e^{-b \cdot t}\left(1 - e^{b \cdot t}\right)^{c-1}$$

In conjunction with volume as the primary variable of interest in the Tyfiant Coed approach another function of the relative increment has been developed based on the Gompertz function:

$$(3.7) \qquad y = a \cdot e^{-b \cdot e^{-c \cdot t}} \text{ with the increment function } y' = a \cdot b \cdot c \cdot e^{-c \cdot t} \cdot e^{-b \cdot e^{-c \cdot t}}$$

In order to compensate for shortcomings of the original Gompertz function in lower age classes Wenk (1969) developed the following equation:

$$(3.8) \qquad p_{i,t+\Delta t} = e^{-c_{1i} \cdot t_i' \left(1 - e^{-c_{2i} \cdot t_i'^{(1 - e^{-c_3 \cdot t_i'})}}\right)}$$

**Definition 3.4.** The parameter $c_1$ in equation (3.8) is the *growth parameter*. At the time of culmination of the current annual increment $c_1$ = relative increment. $c_1 \in [0,1]$.

The parameter $c_2$ accounts for juvenile growth of trees up to an age of 40 years with European species and 25-30 years with Sitka spruce. In most cases the parameters $c_2$ and $c_3$ can be simplified by setting them constant to 1.0 and 0.4, respectively.
The function parameters, especially the growth parameter $c_1$ and the parameter $c_2$ are

interpretable and reflect the vitality of a tree as demonstrated in Pommerening and Wenk (2002).

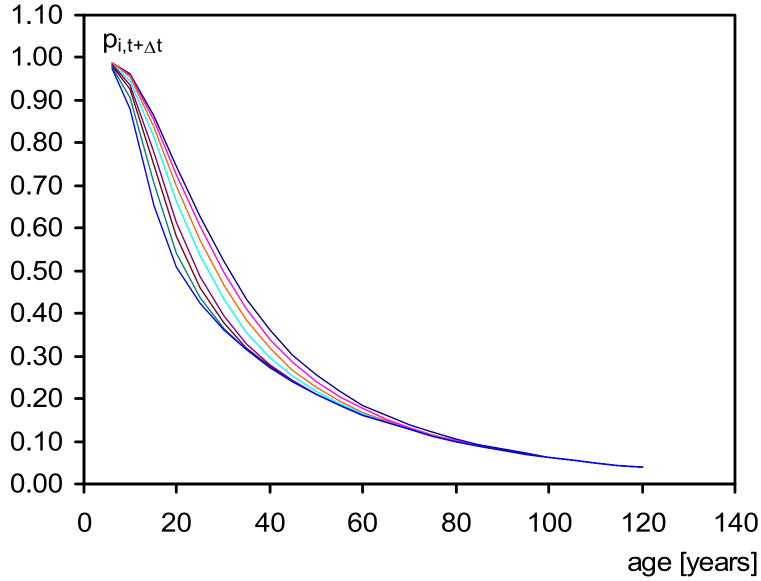


Figure 3.1 The function of the relative increment (3.8) with $c_1 = 0.23$, $c_2$ ranging from 0.5 to 5.0 and $c_3 = 0.4$.

The Wenk function (3.8) always refers to the final age and only gives proper results if calculated for intervals of 10years period. Gerold and Römisch (1977) developed an approximation approach which allows the calculation of $p_{i,t+\Delta t}$ for annual periods. As in most applications where variable interval lengths have to be dealt with this algorithm is central to the modelling theory used and is, therefore, briefly explained here (3.9).

(3.9) $$M_{i,t+1} = \frac{\prod_{j=1}^{n} M_{i,t+j\cdot 10}}{\prod_{j=1}^{n} M_{i,t+j\cdot 10+1}}$$

$M_{i,t+1}$ is the annual growth multiplier. This algorithm only works if the function of the relative increment approaches 0 (zero) with increasing age which function (3.8) does. With increasing $n$ the quotient $M_{i,t+n\cdot 10} / M_{i,t+n\cdot 10+1}$ approaches 1. Therefore $n$ in (3.9) is defined by the point when quotient $M_{i,t+n\cdot 10} / M_{i,t+n\cdot 10+1}$ falls below a certain threshold value, e.g. 1.0001. If observation/forecasting intervals other than 10 or 1 have to be used the correct growth multiplier can be derived from (3.9) by using formula (3.10).

(3.10) $$M_{i,t+n} = \prod_{j=1}^{n} M_{i,t+1}$$

For the purpose of this project algorithms (3.9) and (3.10) were embedded in a DLL written in Delphi (Pascal). The DLL can be embedded in MS EXCEL and MS FoxPro as well as other programming environments. Figures 3.2 and 3.3 illustrate the difference between using the Gerold/Römisch approach (figure 3.2) and assuming that the conventional use of (3.4) and (3.8) lead to correct results for a 4 and 9 year interval. The real growth multipliers were calculated from formula (3.5).

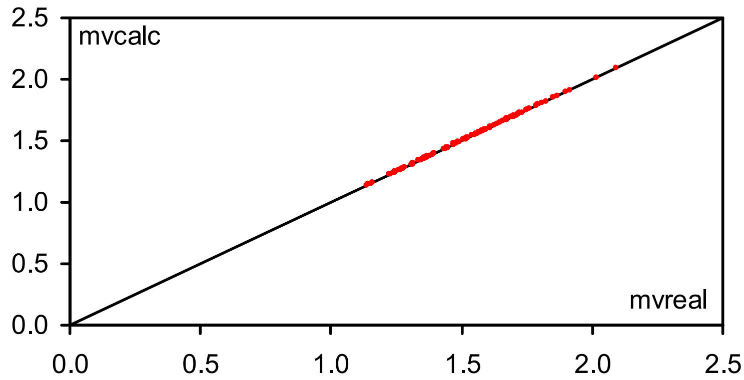


Figure 3.2 Comparison between calculated (using 3.9 and 3.10) growth multiplier (mvcalc) and true growth multiplier (mvreal) at the age of 24 of the SS time series BB 2068 (4 year interval).

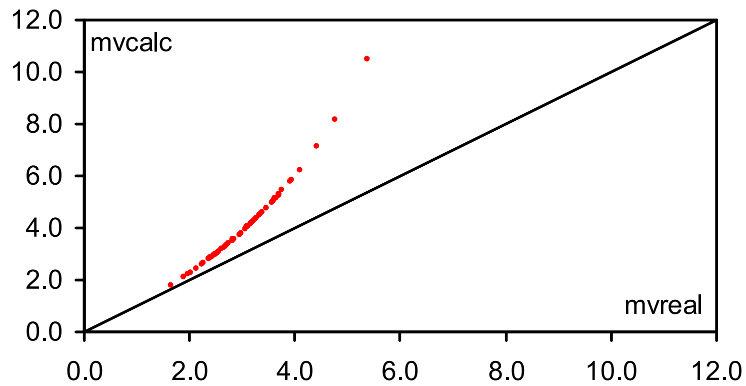


Figure 3.3 Comparison between calculated (ignoring 3.9 and 3.10) growth multiplier (mvcalc) and true growth multiplier (mvreal) at the age of 30 of the SS time series BB 2068 (9 year interval).

The Gerold/Römisch approach leads to a perfect agreement of calculated and observed growth multipliers while deriving the growth multiplier without it results in a bias which increases with increasing growth multiplier. The calculated growth multiplier in this case overestimates the real one.

### 3.3 Allometrics: The modelling of height and diameter growth

The functions developed in 3.1 – 3.2 allow the modelling of volume growth or the growth of biomass. Following the approaches developed by Wenk *et al.* (1990) and Wenk (1994) volume growth is now allometrically linked with height and diameter growth.

As early as the 19$^{th}$ century foresters such as Preßler and Schneider successfully used so-called allometric relationships to explain the growth of one part of a tree by the growth of a different part of the same tree. This is based on the fundamental finding that the relative increment of one growth quantity is proportional to that of another growth quantity of one and the same organism (see also Pienaar and Turnbull, 1973). Allometric relationships are used in this approach to derive height and diameter increment from volume increment.

The volume of a tree at a given time $t$ can be described as the product of basal area $g(t)$, height $h(t)$ and a form factor $f(t)$. The absolute growth values can be replaced by the corresponding multipliers. Then we can derive the reciprocals of all growth multipliers.

(3.11)
$$v(t) = g(t) \cdot h(t) \cdot f(t)$$
$$M_{v,i,t+\Delta t} = M_{h,i,t+\Delta t} \cdot M_{g,i,t+\Delta t} \cdot M_{f,i,t+\Delta t} \qquad |1:$$
$$\Leftrightarrow \quad 1 - p_{v,i,t+\Delta t} = \left(1 - p_{h,i,t+\Delta t}\right) \cdot \left(1 - p_{g,i,t+\Delta t}\right) \cdot \left(1 - p_{f,i,t+\Delta t}\right)$$

The factor $1 - p_{f,i,t+\Delta t}$ can in most cases from an age of 40 years onwards be disregarded because the change of tree form is very marginal. This leads to

(3.12)
$$1 - p_{v,i,t+\Delta t} = \left(1 - p_{h,i,t+\Delta t}\right) \cdot \left(1 - p_{g,i,t+\Delta t}\right)$$

Wenk (1978) found the allometric relationship

$$1 - p_{v,i,t+\Delta t} = \left(1 - p_{h,i,t+\Delta t}\right)^{m_{i,t+\Delta t}},$$

whereby the exponent $m_{i,t+\Delta t}$ is the allometric coefficient. Solved for height multiplier $M_{h,i,t+\Delta t}$ the equation reads

(3.13)
$$M_{h,i,t+\Delta t} = \sqrt[m_{i,t+\Delta t}]{M_{v,i,t+\Delta t}}\ .$$

The diameter multiplier can then be derived as follows Wenk *et al.* (1990, p. 109)

(3.14)
$$1 - p_{d,i,t+\Delta t} = \frac{\left(1 - p_{h,i,t+\Delta t}\right)^{\frac{m-1}{2}}}{\sqrt{1 - p_{f,i,t+\Delta t}}} \qquad |1:$$

(3.15)
$$M_{d,i,t+\Delta t} = \frac{M_{f,i,t+\Delta t}^{-2}}{\left(M_{h,i,t+\Delta t}\right)^{\frac{-m_{i,t+\Delta t}+1}{2}}}$$